Solution 6

1. Show that f is continuous from (X, d) to (Y, ρ) if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y.

   Solution. Use $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ to reduce to the statement: f is continuous iff $f^{-1}(G)$ is open for open G.

2. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}$:

   (a) $[1, 2) \cup (2, 5) \cup \{10\}$.

   (b) $[0, 1] \cap \mathbb{Q}$.

   (c) $\bigcup_{k=1}^{\infty}(1/(k + 1), 1/k)$.

   (d) $\{1, 2, 3, \cdots\}$.

   Solution.

   (a) Boundary points $1, 2, 5, 10$. Interior points $(1, 2), (2, 5)$. Interior $(1, 2)\cup(2, 5)$. Closure $[1, 5] \cup \{10\}$.

   (b) Boundary points: all points in $[0, 1] \cap \mathbb{Q}$. No interior point. Interior ϕ. Closure $[0, 1]$

   (c) Boundary points $\{1/k : \ k \geq 1\}, 0$. Interior points: all points in this set. Interior: This set (because it is an open set). Closure $[0, 1]$.

   (d) Boundary points $1, 2, 3, \cdots$. No interior points. Interior ϕ. Closure: the set itself (it is a closed set).

3. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}^2$:

   (a) $R \equiv [0, 1) \times [2, 3) \cup \{0\} \times (3, 5)$.

   (b) $\{(x, y) : 1 < x^2 + y^2 \leq 9\}$.

   (c) $\mathbb{R}^2 \setminus \{(1, 0), (1/2, 0), (1/3, 0), (1/4, 0), \cdots\}$.

   Solution.

   (a) Boundary points: the geometric boundary of the rectangle and the segment $\{0\} \times [3, 5]$. Interior points: all points inside the rectangle. Interior $(0, 1) \times (3, 5)$. Closure $[0, 1] \times [3, 5] \cup \{0\} \times [3, 5]$.

   (b) Boundary points: all (x, y) satisfying $x^2 + y^2 = 1$ or $x^2 + y^2 = 9$. Interior points: all points satisfying $1 < x^2 + y^2 < 9$. Interior $\{(x, y) : \ 1 < x^2 + y^2 < 9\}$. Closure $\{(x, y) : \ 1 \leq x^2 + y^2 \leq 9\}$.

   (c) Boundary points: The set together with $\{(0, 0)\}$. Interior points: None. Interior ϕ. Closure $\{(0, 0), (1, 0), (1/2, 0), (1/3, 0), \cdots\}$.

4. Describe the closure and interior of the following sets in $C[0, 1]$:

   (a) $\{f : \ f(x) > -1, \ \forall x \in [0, 1]\}$.

   (b) $\{f : \ f(0) = f(1)\}$.

   Solution.

(a) Closure $\{f \in C[0,1] : \ f(x) \geq -1, \ \forall x \in [0,1]\}$. Interior: The set itself. It is an open set.

(b) Closure: The set itself. It is a closed set. Interior: ϕ. For any f satisfying $f(0) = f(1)$, there are many $g \in C[0,1]$ satisfying $\|g - f\|_\infty < \varepsilon$ but $g(0) \neq g(1)$.

5. Let A and B be subsets of (X, d). Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Solution. We have $\overline{A} \subset \overline{B}$ whenever $A \subset B$ right from definition. So $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Conversely, if $x \in \overline{A \cup B}$, $B_\varepsilon(x)$ either has non-empty intersection with A or B. So there exists $\varepsilon_j \to 0$ such that $B_{\varepsilon_j}(x)$ has nonempty intersection with A or B, so $x \in \overline{A} \cup \overline{B}$.

6. Show that $\overline{E} = \{x \in X : \ d(x, E) = 0\}$ for every non-empty $E \subset X$.

Solution. Let $x \in \overline{E}$. By definition, for each n there exists some $y_n \in E$ such that $y_n \in B_{1/n}(x)$. It follows that $d(x, E) \leq d(x, y_n) \to 0$ which implies $d(x, E) = 0$. On the other hand, if $d(x, E) = 0$, there exists $\{x_n\} \subset E$ such that $d(x, x_n) \to 0$, so $x \in \overline{E}$.

7. Show that f is continuous from (X, d) to (Y, ρ) if and only if for every $E \subset X$, $f(\overline{E}) \subset \overline{f(E)}$.

Solution. Let $y_0 = f(x_0)$, $x_0 \in \overline{E}$. We can find $x_n \in E$, $x_n \to x_0$. By continuity, $f(x_n) \to f(x_0) = y_0$. As $f(x_n) \in f(E)$, $y_0 = f(x_0) \in \overline{f(E)}$. Conversely, if for some $x_n \to x_0$ but $f(x_n)$ does not tend to $f(x_0)$, there exists some $B_\rho(f(x_0))$ such that there are infinitely many $f(x_n)$ not belonging to $B_\rho(f(x_0))$. WLOG assume the whole $\{f(x_n)\}$ does, that is, $\{f(x_n)\} \cap B_\rho(f(x_0)) = \phi$ for all n. Now consider the set $F = \{x_1, x_2, \cdots\}$. By assumption, $f(\overline{F}) \subset \overline{f(F)}$. In particular, $f(x_0) \in \overline{f(F)}$, that is, $B_\rho(f(x_0)) \cap \{f(x_n)\} \neq \phi$ for some n, contradiction holds.